Operator Theory and Arithmetic in $H^\infty$

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To the memory of Irina Gorun
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Introduction

The deep relationship between linear algebra and the arithmetical properties of polynomial rings is well understood, and a highlight is naturally Jordan’s classification theorem for linear transformations on a finite-dimensional vector space. The methods and results of finite-dimensional linear algebra seldom extend to, or have analogues in, infinite-dimensional operator theory. Thus it is remarkable to have a class of operators whose properties are closely related with the arithmetic of the ring \( H^\infty \) of bounded analytic functions in the unit disc and for which a classification theorem is available, analogous to Jordan’s classical result. Such a class is the class \( C_0 \), discovered by B. Sz.-Nagy and C. Foiaş in their work on canonical models for contraction operators on Hilbert space. A contraction operator belongs to this class if and only if the associated functional calculus on \( H^\infty \) has a nontrivial kernel. The class \( C_0 \) is the central object of study of this monograph, but we have included other related topics where it seemed appropriate. In an effort to make the book as self-contained as possible we give an introduction to the theory of dilations and functional models for contraction operators (see Chapters 1 and 5). While this introduction is adequate for our purposes, the reader familiar with the basic book [6] by Sz.-Nagy and Foiaş will be able to put the subject matter of this monograph in a greater perspective. Prerequisites for this book are a course in functional analysis (Rudin [2], for instance, will cover most of what we need) and an acquaintance with the theory of Hardy spaces in the unit disc (either Hoffman [1] or Duren [1] covers the required material). In addition, knowledge of the trace class of operators is needed in Chapter 6 (see, for example, Gohberg and Krein [1]).

Quite possibly, the class \( C_0 \) is the best understood class of nonnormal operators on a Hilbert space, even though there are still unsolved problems and unexplored avenues. Besides its intrinsic interest and direct applications, operators of class \( C_0 \) are very helpful as a source of inspiration, and in constructing examples and counterexamples in other branches of operator theory. Interestingly, the class \( C_0 \) also surfaces in certain problems of control and realization theory. It is hoped that this book will be interesting for operator theorists (present or to be), as well as those theoretical engineers who are interested in the applications of operator theory.
I tried to make this book more useful by including a number of exercises for each section. The numbering of theorems, propositions, etc. is conceived such as to make cross-references easy. For instance, Theorem 8.1.8 is in §1 of Chapter 8, and it is followed by relation (8.1.9) and Lemma 8.1.10. The first numeral is omitted for references within the same chapter. Each chapter begins with a description of the material to be covered. References to the literature and historical comments are kept to a minimum in the text. There is an appendix dedicated to these questions.

My teachers, colleagues, and friends Ciprian Foiaș, Carl Pearcy, Béla Sz.-Nagy, and Dan Voiculescu encouraged me at various times to write this book. Part of the book or earlier versions of some chapters were written while I was at the University of Michigan, the Massachusetts Institute of Technology, the Mathematical Sciences Research Institute, and Indiana University. Much of the material was presented in a seminar at the University of Michigan. I am grateful to all of these institutions for their hospitality and to some of them for help in typing the manuscript.

My wife Irina, with her exceptional talent and warmth, has been an inspiration for me during most of my mathematical life. Irina helped me get through difficult times and gave me determination and ambition when I lacked them. This book is dedicated to her memory.

Hari Bercovici
CHAPTER 1

An Introduction to Dilation Theory

Any contraction, i.e., operator of norm \( \leq 1 \), on a Hilbert space has a unitary dilation. This is Sz.-Nagy’s theorem, and it was the starting point of an important branch in operator theory. In this chapter we give the basic elements of dilation theory, which will help us enter the subject proper of the book in Chapter 2. In Section 1 we present Sz.-Nagy’s dilation theorem mentioned above. As a consequence we deduce the decomposition of any contraction into a direct sum of unitary and completely nonunitary parts. We also give a proof of the commutant lifting theorem, which relates the commutant of a contraction with the commutants of its isometric and unitary dilations. Section 2 contains more detailed information about the minimal isometric dilation of an operator. It is shown that the completely nonunitary summand of an isometry is a unilateral shift, and conditions are given on an operator which ensure that its minimal isometric dilation is a unilateral shift. An important result concerns the absolute continuity (with respect to Lebesgue arclength measure on the unit circle) of the minimal unitary dilation. In Section 3 we discuss the notions of cyclic multiplicity, quasisimilarity, and quasiaffine transforms. The latter two notions are weak forms of similarity. The most important result (Theorem 3.7) relates an operator \( T \), with small cyclic multiplicity, to a simpler operator. This result is the starting point of the classification theory of operators of class \( C_0 \).

1. Unitary dilations of contractions. Let \( T \) be a contraction on the Hilbert space \( \mathcal{H} \). We will use the following notation:

\[
\begin{align*}
D_T &= (I - T^*T)^{1/2}, & D_{T^*} &= (I - TT^*)^{1/2}, \\
\mathcal{D}_T &= (\text{ran } D_T)^-, & \mathcal{D}_{T^*} &= (\text{ran } D_{T^*})^-.
\end{align*}
\]

The operator \( D_T \) is called the defect operator of \( T \) and \( \mathcal{D}_T \) the defect space. Using the functional calculus for selfadjoint operators, it is easy to see that the obvious relation \( T(I - T^*T) = (I - TT^*)T \) implies

\[TD_T = D_{T^*}T.\]

In particular, we have \( T\mathcal{D}_T \subset \mathcal{D}_{T^*} \).

Easier to understand among contractions are the isometric and unitary operators. Arbitrary contractions can be related to isometries using dilations. We
recall that if $\mathcal{H}$ is a Hilbert space, $\mathcal{H} \subset \mathcal{H}$ is a subspace, $S \in \mathcal{L}(\mathcal{H})$, and $T \in \mathcal{L}(\mathcal{H})$, then $S$ is a dilation of $T$ (and $T$ is a power-compression of $S$) provided that

$$T^n = P_{\mathcal{H}} S^n |_{\mathcal{H}}, \quad n = 0, 1, 2, \ldots.$$

If, in addition, $S$ is an isometry (unitary operator) then $S$ will be called an isometric (unitary) dilation of $T$. An isometric (unitary) dilation $S$ of $T$ is said to be minimal if no restriction of $S$ to an invariant subspace is an isometric (unitary) dilation of $T$. The following result is left as an exercise.

1.3. LEMMA. Let $S$ be an isometric (unitary) dilation of $T$. Then $S$ is a minimal isometric (unitary) dilation of $T$ if and only if $\bigvee_{n=0}^{\infty} S^n \mathcal{H} = \mathcal{H}$ ($\bigvee_{n=-\infty}^{\infty} S^n \mathcal{H} = \mathcal{H}$).

The proof of the next result is motivated by the following calculation:

$$||x||^2 - ||Tx||^2 = (x, x) - (T^* Tx, x) = ||D_T x||^2,$$

$x \in \mathcal{H}$,

which shows that the operator $X : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ defined by $X x = Tx \oplus D_T x$ is isometric. Of course, $X$ is not a dilation of $T$ because it acts between two different Hilbert spaces.

1.4. THEOREM. Every contraction $T \in \mathcal{L}(\mathcal{H})$ has a minimal isometric dilation. This dilation is unique in the following sense: if $S \in \mathcal{L}(\mathcal{H})$ and $S' \in \mathcal{L}(\mathcal{H}')$ are two minimal isometric dilations for $T$, then there exists an isometry $U$ of $\mathcal{H}$ onto $\mathcal{H}'$ such that $U x = x$, $x \in \mathcal{H}$, and $S' U = U S$.

PROOF. We first prove the uniqueness part. Thus, let $S \in \mathcal{L}(\mathcal{H})$ and $S' \in \mathcal{L}(\mathcal{H}')$ be two minimal isometric dilations of $T$ and note that, by Lemma 1.3, we have

$$\mathcal{H} = \bigvee_{n=0}^{\infty} S^n \mathcal{H}, \quad \mathcal{H}' = \bigvee_{n=0}^{\infty} S'^n \mathcal{H}.$$ 

If $\{x_j\}_{j=0}^{\infty}$ is a finitely nonzero family of vectors in $\mathcal{H}$, we have

$$\left|\sum_{j=0}^{\infty} S^j x_j \right|^2 = \sum_{j,k=0}^{\infty} (S^j x_j, S^k x_k).$$

Since $S$ is an isometry, we have $(S^j x_j, S^k x_k) = (S^{j'} x_j, S^{k'} x_k)$ if $k - j = k' - j'$ and therefore

$$\left|\sum_{j=0}^{\infty} S^j x_j \right|^2 = \sum_{j \geq k} (S^{j-k} x_j, x_k) + \sum_{j < k} (x_j, S^{k-j} x_k)$$

$$= \sum_{j \geq k} (S^{j-k} x_j, P_{\mathcal{H}} x_k) + \sum_{j < k} (P_{\mathcal{H}} x_j, S^{k-j} x_k)$$

$$= \sum_{j \geq k} (P_{\mathcal{H}} S^{j-k} x_j, x_k) + \sum_{j < k} (x_j, P_{\mathcal{H}} S^{k-j} x_k)$$

$$= \sum_{j \geq k} (T^{j-k} x_j, x_k) + \sum_{j < k} (x_j, T^{k-j} x_k),$$
where we used the fact that $S$ is a power-dilation of $T$. A similar computation for $S'$ shows that $\|\sum_{j=0}^{\infty} S^j x_j \| = \|\sum_{j=0}^{\infty} S'^j x_j \|$. This easily implies the existence of an isometry $U$ of $\mathcal{H}$ onto $\mathcal{H}'$ satisfying

$$U \left( \sum_{j=0}^{\infty} S^j x_j \right) = \sum_{j=0}^{\infty} S'^j x_j$$

for every finitely nonzero sequence $\{x_j\}_{j=0}^{\infty}$ in $\mathcal{H}$. Clearly then $Ux = x$, $x \in \mathcal{H}$, and $S'U = US$, so that uniqueness is proved.

For the existence part, we define the space $\mathcal{H}_+$ by

$$\mathcal{H}_+ = \mathcal{H} \oplus \left( \bigoplus_{n=0}^{\infty} \mathcal{D}_n \right), \quad \mathcal{D}_n = \mathcal{D}_T, \ n = 0, 1, 2, \ldots,$$

and the operator $U_+ \in \mathcal{L}(\mathcal{H}_+)$ by

$$U_+ \left( x \oplus \left( \bigoplus_{n=0}^{\infty} d_n \right) \right) = Tx \oplus \left( \bigoplus_{n=0}^{\infty} e_n \right)$$

where $e_0 = D_T x$ and $e_n = d_{n-1}$, $n \geq 1$. Since $\|Tx\|^2 + \|D_T x\|^2 = \|x\|^2$, it is obvious that $U_+$ is an isometry. It is also clear that $U_+$ is an isometric dilation of $T$, if we identify the vector $x \in \mathcal{H}$ with the vector $x \oplus (\bigoplus_{n=0}^{\infty} 0) \in \mathcal{H}$; in fact $\mathcal{H}$ is invariant under $U_+^*$ and $T^* = U_+^* | \mathcal{H}$. It remains to be shown that $U_+$ is minimal. It is clear that $\mathcal{H} \vee U_+ \mathcal{H}$ contains all elements of the form $0 \oplus D_T x \oplus 0 \oplus \cdots, x \in \mathcal{H}$, so that

$$\mathcal{H} \vee U_+ \mathcal{H} = \mathcal{H} \oplus \mathcal{D}_T \oplus \{0\} \oplus \cdots.$$

It now follows from the definition of $U_+$ that

$$\bigvee_{j=0}^{n} U_+^j \mathcal{H} = \mathcal{H} \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \cdots \oplus \mathcal{D}_T \oplus \{0\} \oplus \cdots$$

and the minimality of $U_+$ follows from Lemma 1.3.

The following result is the counterpart of Theorem 1.4 for unitary dilations.

1.5. THEOREM. Every contraction $T \in \mathcal{L}(\mathcal{H})$ has a minimal unitary dilation, unique in the sense specified in Theorem 1.4.

PROOF. The uniqueness is proved using the same calculation as in the proof of Theorem 1.4, except that one would consider sums of the form $\sum_{j=-\infty}^{\infty} S^j x_j$, $x_j \in \mathcal{H}$. In order to prove the existence of a minimal unitary dilation we consider the space $\mathcal{H}$ defined as

$$\mathcal{H} = \left( \bigoplus_{j=-\infty}^{0} \mathcal{E}_j \right) \oplus \mathcal{H} \oplus \left( \bigoplus_{j=0}^{\infty} \mathcal{D}_j \right),$$
where $\mathcal{E}_{-j} = \mathcal{D}_T$, and $\mathcal{D}_j = \mathcal{D}_T$, $j = 0, 1, 2, \ldots$, and the operator $U \in \mathcal{L}(\mathcal{H})$ defined by

$$U \left( \left( \bigoplus_{j=-\infty}^{0} e_j \right) \oplus x \oplus \left( \bigoplus_{j=0}^{\infty} d_j \right) \right) = \left( \bigoplus_{j=-\infty}^{0} e'_j \right) \oplus x' \oplus \left( \bigoplus_{j=0}^{\infty} d'_j \right),$$

where $x' = Tx + DT_e e_0$, $d'_0 = -T^* e_0 + DT_e x$, $d'_j = d_{j-1}$, $j \geq 1$, and $e'_j = e_{j-1}$, $j \leq 0$. The space $\mathcal{H}_+$, constructed in the previous proof, can be identified with $\{0\} \oplus \mathcal{H}_+ \subset \mathcal{H}$, and clearly $U_+ = U \mid \mathcal{H}_+$. It follows at once that $U$ becomes a dilation of $T$ upon the identification of $\mathcal{H}$ with $\{0\} \oplus \mathcal{H} \oplus \{0\} \subset \mathcal{H}$. In order to show that $U$ is unitary it suffices to show that $U$ and $U^*$ are isometries. The fact that $U$ is an isometry is equivalent to the identity

$$||Tx + DT_e e_0||^2 + ||-T^* e_0 + DT_e x||^2 = ||e_0||^2 + ||x||^2, \quad e_0 \in \mathcal{D}_T, \quad x \in \mathcal{H}.$$ 

The left-hand side of this identity can be rewritten as follows:

$$||Tx||^2 + ||DT_e e_0||^2 + 2 \Re(Tx, DT_e e_0) + ||T^* e_0||^2$$

$$+ ||DT_e x||^2 - 2 \Re(DT_e x, T^* e_0)$$

$$= ||x||^2 + ||e_0||^2 + 2 \Re[(x, T^* DT_e e_0) - (x, DT_e T^* e_0)],$$

and the required identity follows from (1.2) applied to $T^*$. The minimality of $U$ and the fact that $U^*$ is also an isometry are left as exercises.

As noted above, the space $\mathcal{H}_+$ constructed in the proof of Theorem 1.4 can (and shall) be considered as a subspace of $\mathcal{H}$, invariant under $U$:

$$U_+ = U \mid \mathcal{H}_+.$$ 

Thus $U$ is also a minimal unitary dilation of $U_+$. In fact $U^*$ is the minimal isometric dilation of $U_+^*$ and therefore a different proof of Theorem 1.5 would consist in showing that the minimal isometric dilation of an operator of the form $U_+^*$ is always unitary. We chose the above proof because it is more difficult to identify the defect space of $U_+^*$ in terms of the original operator $T$.

1.6. DEFINITION. A contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be completely nonunitary if there is no invariant subspace $\mathcal{M}$ for $T$ such that $T \mid \mathcal{M}$ is a unitary operator.

An important consequence of Theorem 1.5 is the following.

1.7. PROPOSITION. For every contraction $T \in \mathcal{L}(\mathcal{H})$ there exist reducing subspaces $\mathcal{H}_0, \mathcal{H}_1$ for $T$ such that

(i) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$;

(ii) $T \mid \mathcal{H}_1$ is completely nonunitary; and

(iii) $T \mid \mathcal{H}_0$ is a unitary operator.

The spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are uniquely determined by conditions (i)–(iii).

PROOF. Let $U \in \mathcal{L}(\mathcal{H})$ be a minimal unitary dilation of $T$. Denote by $\mathcal{H}_0$ the reducing subspace for $U$ generated by $\mathcal{H} \ominus \mathcal{H}$ and set $\mathcal{H}_0 = \mathcal{H} \ominus \mathcal{H}_0$. Obviously $\mathcal{H}_0 \subset \mathcal{H}$ and $Tx = Ux, x \in \mathcal{H}_0$, because $\mathcal{H}_0$ reduces $U$. Thus $\mathcal{H}_0$ is
reducing for $T$ and $T \mid \mathcal{H}_0 = U \mid \mathcal{H}_0$ is unitary. We now set $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$ and prove that $T \mid \mathcal{H}_1$ is completely nonunitary. If $\mathcal{M} \subset \mathcal{H}$ is invariant for $T$ and $T \mid \mathcal{M}$ is unitary then the equalities

$$||h|| = ||Th|| = ||P_{\mathcal{M}}Uh||, \quad h \in \mathcal{M},$$

imply that $Th = Uh$ for $h \in \mathcal{M}$. Thus $\mathcal{M}$ is invariant for $U$, $U \mid \mathcal{M}$ is unitary, and hence $\mathcal{M}$ is reducing for $U$. Now, $\mathcal{M}$ is orthogonal onto $\mathcal{H} \ominus \mathcal{H}_0$ and therefore onto $\mathcal{H}_0$; we deduce that $\mathcal{M} \subset \mathcal{H}_0$. This argument shows at once that $T \mid \mathcal{H}_1$ is completely nonunitary and that the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is unique with the properties (i)-(iii).

The preceding result shows that the study of general contractions can be reduced in many cases to the study of the completely nonunitary ones.

Before proving an important property of isometric and unitary dilations we study in further detail the space of the minimal isometric dilation of a contraction $T \in \mathcal{L}(\mathcal{H})$. Let us denote by $\mathcal{H}_n$, $n = 0, 1, 2, \ldots$, the subspace of $\mathcal{H}_+$ defined as

$$\mathcal{H}_n = \mathcal{H} \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \cdots \oplus \mathcal{D}_T \oplus \{0\} \oplus \cdots$$

Thus $\mathcal{H}_0 = \mathcal{H}$ and each $\mathcal{H}_n$ is invariant for $U_+^*$. If we set $T_n = P_{\mathcal{H}_n}U_+ \mid \mathcal{H}_n$, then $T_{n+1}$ is a dilation of $T_n$ for every $n$. The contractions $T_n$ can be viewed differently. For an arbitrary contraction $S \in \mathcal{L}(\mathcal{H})$ we can construct a dilation $S_\sim$ of $S$ on $\mathcal{H} \oplus \mathcal{D}_S$ defined by

$$(1.8) \quad S_\sim(x \oplus y) = Sx \oplus DSx.$$ 

Clearly then $S_\sim$ is a partial isometry and

$$\mathcal{D}_{S_\sim} = \ker S_\sim = \{0\} \oplus \mathcal{D}_S.$$ 

Thus if we repeat this procedure, we can construct a partial isometry $S_{\sim \sim} = (S_\sim)_\sim$ which dilates $S_\sim$, acts on $\mathcal{H} \oplus \mathcal{D}_S \oplus \mathcal{D}_S$, and is defined by

$$S_{\sim \sim}(x \oplus y \oplus z) = Sx \oplus DSx \oplus y.$$ 

It is clear now that the contractions $T_n$ considered above satisfy the relations

$$T_{n+1} = (T_n)_\sim, \quad n = 0, 1, 2, \ldots,$$

up to natural unitary equivalences.

1.9. PROPOSITION. Let $T \in \mathcal{L}(\mathcal{H})$ and $T' \in \mathcal{L}(\mathcal{H}')$ be two contractions, and let $X \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ satisfy the intertwining relation $T'X = XT$. Then there exists an operator $Y \in \mathcal{L}(\mathcal{H} \oplus \mathcal{D}_T, \mathcal{H}' \oplus \mathcal{D}_{T'})$ such that

(i) $Y(\{0\} \oplus \mathcal{D}_T) \subset \{0\} \oplus \mathcal{D}_{T'}$;

(ii) $P_{\mathcal{H}'}Y \mid \mathcal{H} = X$;

(iii) $||Y|| = ||X||$; and

(iv) $T'_\sim Y = YT_\sim$, where $T_\sim$ and $T'_\sim$ are the dilations of $T$ and $T'$ described by (1.8).